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On a class of Laplace inverses involving doubly-nested square roots and their applications in continuum mechanics

P. Puri [a,1], P.M. Jordan [b,*]

[a] Dept. of Mathematics, University of New Orleans, New Orleans, LA 70148, USA
[b] Code 7181, Naval Research Laboratory, Stennis Space Center, MS 39529, USA

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ABSTRACT

The exact inverses of a class of Laplace transforms involving doubly-nested square roots and two branch points are presented. From these results the exact solution of Stokes' first problem, a well known initial-boundary value problems (IBVP) in fluid mechanics, for a micropolar fluid is determined.

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1. Introduction and problem formulation

Using the Laplace transform, we (Jordan and Puri, 1999) derived the exact solution to Stokes' first problem for a dipolar fluid (see Cowin (1974), Straughan (1987) and the references therein) that is valid for arbitrary values of the material parameters $d(\geqslant 0)$ and $l(>0)$. The present Note is devoted to extending this earlier work of ours to the case of micropolar fluids, the theory of which was formulated by Eringen (1966). In doing so, we also extend the results of Jordan et al. (2000) on Laplace inverses involving doubly-nested square roots with one branch point to a more general class of such inverses possessing two branch points.

To this end, we observe that by replacing the straight channel of Kirwan and Newman (1972),[2] who considered the unsteady plane Couette flow of an incompressible micropolar fluid, with the half-space $y > 0$, and setting the pressure gradient term in their equation of motion to zero, one possible way to formulation Stokes' first problem for a micropolar fluid is as the IBVP

$$\rho \mathscr{I} u_{tt} + 2k u_t - k(2\nu + \kappa) u_{yy} - [\gamma + \mathscr{I}(\mu + k)] u_{yyt} + \gamma(\nu + \kappa) u_{yyyy} = 0, \quad (y,t) \in (0,\infty) \times (0,\infty), \tag{1.1}$$

$$u(0,t) = U_0 \theta(t), \ u(\infty,t) = 0, \ u_{yy}(0,t) = M_s, \ u_{yy}(\infty,t) = 0 \ (t > 0), \tag{1.2}$$

$$u(y,0) = u_t(y,0) = 0 \quad (y > 0). \tag{1.3}$$

In IBVP (1), the velocity vector has the form $\mathbf{v} = (u(y,t), 0, 0)$, where the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ is identically satisfied; $\theta(\cdot)$ denotes the Heaviside unit step function; $\mathscr{I}$, γ, k, and the dynamic viscosity μ are positive constants; here, $\nu = \mu/\rho$ denotes the kinematic viscosity; U_0 is a nonzero constant; M_s is taken as constant; and we have set $\kappa := k/\rho$ for convenience.

Remark 1. The boundary conditions (BC)s assumed in Eq. (1.2), which in the present context are equivalent to BCs of "type A" in dipolar fluid theory (Straughan, 1987), were suggested by Kirwan and Newman (1972). For more on the question of what BCs are appropriate in the micropolar context, see also Eringen (1966) and Cowin (1974).

2. Exact solution via the Laplace transform

To reduce the number of coefficients, we introduce the following nondimensional quantities: $u' = u/U_0$, $y' = y/L_0$, $t' = t/T_0$, where

$$L_0 = \sqrt{\frac{\gamma + \mathscr{I}(2\mu + k)}{2k}} \qquad \text{and} \qquad T_0 = \frac{2L_0^2}{2\nu + \kappa}, \tag{2}$$

and recast IBVP (1) in the simpler form

$$h u_{tt} + u_t - u_{yy} - u_{yyt} + \ell^2 u_{yyyy} = 0, \quad (y,t) \in (0,\infty) \times (0,\infty), \tag{3.1}$$

$$u(0,t) = \theta(t), \quad u(\infty,t) = 0, \quad u_{yy}(0,t) = M, \quad u_{yy}(\infty,t) = 0 \quad (t > 0), \tag{3.2}$$

$$u(y,0) = u_t(y,0) = 0 \quad (y > 0), \tag{3.3}$$

where M is the nondimensional form of M_s and all primes have been suppressed but remain understood. Furthermore, $\ell > 0$ and $h \in (0, 1/2)$ are given by

$$\ell = \frac{1}{L_0}\sqrt{\frac{\gamma(\mu + k)}{k(2\mu + k)}}, \qquad h = \frac{\mathscr{I}(2\mu + k)}{4kL_0^2}. \tag{4}$$

On applying the temporal Laplace transform, $\mathscr{L}[\cdot]$, to Eqs. (3.1) and (3.2), using the initial conditions given in Eq. (3.3), and then solving the resulting subsidiary equation, the transform domain solution is found to be

$$\bar{u}(y, s) = \frac{\ell^2[e^{-r_2 y}r_1^2 - e^{-r_1 y}r_2^2 - M(e^{-r_2 y} - e^{-r_1 y})]}{s\sqrt{s^2(1 - 4h\ell^2) + 2s(1 - 2\ell^2) + 1}}, \tag{5}$$

where s is the transform parameter, a bar over a quantity denotes the image of that quantity in the Laplace transform domain (e.g., $\bar{u}(y, s) := \mathscr{L}[u(y, t)]$), and

$$r_{1,2} = \frac{1}{\ell}\sqrt{\frac{s + 1 \pm \sqrt{(s+1)^2 - 4\ell^2 s(hs + 1)}}{2}}. \tag{6}$$

To simplify the task of inversion, it is convenient to recast Eq. (5) as

$$\bar{u}(y, s) = \bar{u}_1(y, s) - (2\ell^2 M - 1)\bar{u}_2(y, s) + \bar{u}_3(y, s), \tag{7}$$

where

$$\bar{u}_1 = \frac{e^{-r_2 y} + e^{-r_1 y}}{2s}, \quad \bar{u}_2 = \frac{e^{-r_2 y} - e^{-r_1 y}}{2s\sqrt{(s+1)^2 - 4\ell^2 s(hs + 1)}},$$
$$\bar{u}_3 = s\bar{u}_2. \tag{8}$$

Since the inverses we require do not appear to be given in any of the standard tables, we turn to the complex inversion formula (see, e.g., Duffy, 2004) and observe that the inverses of the terms in Eq. (7) are given by

$$u_{1,2,3}(y, t) = \frac{1}{2\pi i}\lim_{T \to \infty}\int_{\sigma - iT}^{\sigma + iT} e^{st}\bar{u}_{1,2,3}(y, s)\,ds \qquad (t > 0), \tag{9}$$

where $u_{1,2,3}(y, t) = 0$ for $t < 0$ and the arbitrary real number $\sigma > 0$ lies to the right of all singularities. The first step in evaluating these integrals is determining the singularities of their integrands. Omitting the details, it can be shown that the only non-removable singularities of $\bar{u}_{1,2}$ consist of a simple pole at $s = 0$ and branch points at $s = \{-1/h, 0\}$ while those of $\bar{u}_3$ consist of only the two branch points. Thus, by Cauchy's theorem we have

$$\frac{1}{2\pi i}\oint_C e^{st}\bar{u}_{1,2,3}(y, s)\,ds = 0, \tag{10}$$

where the Bromwich contour C (see Fig. 1) is traversed in the counterclockwise direction.

Following standard techniques of contour integration (see, e.g., Duffy, 2004), it can be established that, for all $h \in (0, 1/2)$ and $\ell > 0$,

$$u_1(y,t) = \theta(t)\left\{\frac{1}{2}(1 + e^{-y/\ell}) - \frac{1}{2\pi}\int_0^{1/h}\frac{e^{-\eta t}\sin[y\omega(\eta)]d\eta}{\eta}\right\}$$
$$-\frac{\theta(t)}{\pi}\begin{cases} \int_0^\infty \dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_+(\eta)]\cos[y\omega_-(\eta)]d\eta}{\eta+1/h}, & \ell \leqslant (4h)^{-1/2}, \\[4mm] \int_0^{\eta^*}\dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_+(\eta)]\cos[y\omega_-(\eta)]d\eta}{\eta+1/h}, \\[4mm] +\int_{\eta^*}^\infty\dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_+(\eta)]\cosh[y\omega_-(\eta)]d\eta}{\eta+1/h}, & \ell > (4h)^{-1/2}, \end{cases}$$
$$\tag{11}$$

$$u_2(y,t) = \theta(t)\left\{\frac{1}{2}(1 - e^{-y/\ell}) - \frac{1}{2\pi}\int_0^{1/h}\frac{e^{-\eta t}\sin[y\omega(\eta)]d\eta}{\eta\sqrt{(\eta-1)^2 + 4\eta\ell^2(1-h\eta)}}\right\}$$
$$-\frac{\theta(t)}{\pi}\begin{cases} \int_0^\infty \dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_-(\eta)]\cos[y\omega_+(\eta)]d\eta}{(\eta+1/h)\sqrt{[(\eta+1/h)-1]^2 - 4\eta\ell^2(h\eta+1)}}, & \ell \leqslant (4h)^{-1/2}, \\[4mm] \int_0^{\eta^*}\dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_-(\eta)]\cos[y\omega_+(\eta)]d\eta}{(\eta+1/h)\sqrt{[(\eta+1/h)-1]^2 - 4\eta\ell^2(h\eta+1)}}, \\[4mm] +\int_{\eta^*}^\infty\dfrac{\exp[-t(\eta+1/h)]\sinh[y\omega_-(\eta)]\cos[y\omega_+(\eta)]d\eta}{(\eta+1/h)\sqrt{|[(\eta+1/h)-1]^2 - 4\eta\ell^2(h\eta+1)|}}, & \ell > (4h)^{-1/2}, \end{cases}$$
$$\tag{12}$$

$$u_3(y,t) = \theta(t)\left\{\frac{1}{2\pi}\int_0^{1/h}\frac{e^{-\eta t}\sin[y\omega(\eta)]d\eta}{\sqrt{(\eta-1)^2 + 4\eta\ell^2(1-h\eta)}}\right\}$$
$$+\frac{\theta(t)}{\pi}\begin{cases} \int_0^\infty \dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_-(\eta)]\cos[y\omega_+(\eta)]d\eta}{\sqrt{[(\eta+1/h)-1]^2 - 4\eta\ell^2(h\eta+1)}}, & \ell \leqslant (4h)^{-1/2}, \\[4mm] \int_0^{\eta^*}\dfrac{\exp[-t(\eta+1/h)]\sin[y\omega_-(\eta)]\cos[y\omega_+(\eta)]d\eta}{\sqrt{[(\eta+1/h)-1]^2 - 4\eta\ell^2(h\eta+1)}}, \\[4mm] +\int_{\eta^*}^\infty\dfrac{\exp[-t(\eta+1/h)]\sinh[y\omega_-(\eta)]\cos[y\omega_+(\eta)]d\eta}{\sqrt{|[(\eta+1/h)-1]^2 - 4\eta\ell^2(h\eta+1)|}}, & \ell > (4h)^{-1/2}; \end{cases}$$
$$\tag{13}$$

where

$$\omega(\eta) = \frac{1}{\ell}\sqrt{\frac{\eta - 1 + \sqrt{(\eta-1)^2 + 4h\ell^2\eta(1/h - \eta)}}{2}}, \tag{14}$$

$$\omega_\pm(\eta) = \frac{1}{2\ell}\sqrt{\left|(\eta + 1/h) - 1 \pm 2\ell\sqrt{h\eta(\eta + 1/h)}\right|}, \tag{15}$$

$$\eta^* = \frac{1 - h(1 + 2\ell^2 - 2\ell\sqrt{\ell^2 + h - 1})}{h(4h\ell^2 - 1)}, \tag{16}$$

and we observe that $\eta^* > 0$ if and only if $\ell > 1/\sqrt{4h}$.
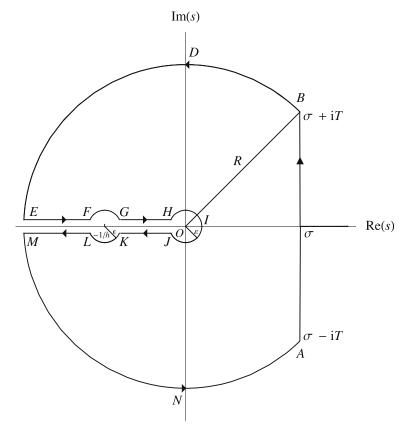
3. Special/limiting cases and other applications

Remark 2. The steady-state solution, $u_\infty(y)$, is defined as $u_\infty(y) := \lim_{t \to \infty}u(y, t)$ and is given by $u_\infty(y) := 1 - M\ell^2[1 - \exp(-y/\ell)]$, which is identical in form to that of the dipolar fluid case (see Jordan and Puri, 1999, Eq. (4.3)).

Remark 3. If we take $h = 1 - \ell^2$, implying $\ell^2 \in (\frac{1}{2}, 1)$, then $\bar{u}_{1,2,3}$ simplify to such an extent that they can be easily inverted using, e.g., the tables of Laplace inverses given in (Erdélyi, 1954, Chap. V). Thus, for this special case $u(y, t) = \mathscr{U}(y, t)$, where $\mathscr{U}$ is defined as

$$\mathscr{U}(y, t) := \theta(t)\left\{\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) - \frac{a\ell^2}{2}e^{at}\left[e^{y\sqrt{a}}\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{at}\right)\right.\right.$$
$$+ e^{-y\sqrt{a}}\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{at}\right) - e^{cy\sqrt{a+b}}\mathrm{erfc}\left(\frac{cy}{2\sqrt{t}} + \sqrt{(a+b)t}\right)$$
$$- e^{-cy\sqrt{a+b}}\mathrm{erfc}\left(\frac{cy}{2\sqrt{t}} - \sqrt{(a+b)t}\right)\Bigg]$$
$$- M\ell^2\left[\mathrm{erfc}\left(\frac{y}{2\sqrt{t}}\right) - \frac{1}{2}e^{at}\left[e^{y\sqrt{a}}\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{at}\right)\right.\right.$$
$$+ e^{-y\sqrt{a}}\mathrm{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{at}\right) - e^{cy\sqrt{a+b}}\mathrm{erfc}\left(\frac{cy}{2\sqrt{t}} + \sqrt{(a+b)t}\right)$$
$$- e^{-cy\sqrt{a+b}}\mathrm{erfc}\left(\frac{cy}{2\sqrt{t}} - \sqrt{(a+b)t}\right)\Bigg] - \frac{1}{2}\left[e^{cy\sqrt{b}}\mathrm{erfc}\left(\frac{cy}{2\sqrt{t}} + \sqrt{bt}\right)\right.$$
$$+ e^{-cy\sqrt{b}}\mathrm{erfc}\left(\frac{cy}{2\sqrt{t}} - \sqrt{bt}\right)\Bigg]\Bigg]\Bigg\}. \tag{17}$$

Fig. 1. Bromwich contour, C, used to evaluate the integrals in Eq. (10). Here, ϵ and ε are small positive quantities.

Here, erfc $(\cdot)$ denotes the complementary error function and the positive constants a, b, and c are defined as $a = (2\ell^2 - 1)^{-1}$, $b = (1 - \ell^2)^{-1}$, and $c = \ell^{-1}\sqrt{1 - \ell^2}$.

Remark 4. Setting $\ell = l_2$ and letting $h \to 0$ reduces u to the $l_1 = 1$ special case of (Jordan and Puri, 1999, Eq. (3.4)), i.e., the corresponding solution for the case of a dipolar[3] fluid. On the other hand, letting $\ell \to 0$ reduces u to the solution of Stokes' first problem for an Oldroyd-B liquid; see pp. 354–355 of Duffy (2004), and also Christov and Jordan (2009).

Remark 5. It is noteworthy that Eq. (1.1), which is identical in form to the equation of motion for this problem involving a microstructure-laden fluid of the type described by the theory of Kline and Allen (1970), also arises in the study of incompressible flow in fractured or fissured media (Guenther and Lee, 1996). Eq. (1.1) can also be regarded as the governing equation for a particular type of two-species reaction–diffusion system.

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[3] In the context of the present problem, dipolar fluid theory and Green–Naghdi theory yield, allowing for differences in the coefficients, the same equation of motion; see, e.g., Quintanilla and Straughan (2005) and the references therein.